Week 4

Q1: Prove that $\mathcal{L}(V, W)$ is a vector space under Def. 3.6.

Proof:

(VS1) Commutativity $\forall f_1, f_2 \in \mathcal{L}(V, W)$,

$\forall v \in V$, $(f_1 + f_2)(v) = f_1(v) + f_2(v) = f_2(v) + f_1(v)$

$= (f_2 + f_1)(v) \Rightarrow f_1 + f_2 = f_2 + f_1$.

(VS2) Additive Associativity Similar proof with (VS1)

(VS3) Additive Identity The zero mapping

$T_0 : V \to W$ with $T_0(v) = 0_W$, $\forall v \in V$

is the additive identity of $\mathcal{L}(V, W)$

(VS4) Additive Inverse $\forall f \in \mathcal{L}(V, W)$,

then $-f \in \mathcal{L}(V, W)$, where $(-f)(v) = -f(v)$, $\forall v \in V$.

then $f + (-f) = T_0$

(VS5) Multiplicative Identity $\forall f \in \mathcal{L}(V, W)$,

$(1 \cdot f)(v) = 1 \cdot f(v) = f(v)$, $\forall v \in V$

$\Rightarrow 1 \cdot f = f$

(VS6) Multiplicative Associativity $\forall \lambda, \mu \in \mathbb{F}$, $\forall f \in \mathcal{L}(V, W)$,

$\forall v \in V$, $((\lambda\mu) f)(v) = (\lambda\mu) \cdot f(v) = \lambda \cdot (\mu \cdot f(v)) = \lambda((\mu f)(v))$

$= (\lambda(\mu f))(v) \Rightarrow (\lambda\mu) f = \lambda(\mu f)$

(VS7) Distributive Properties.

(i). $\forall \lambda \in \mathbb{F}$, $\forall f_1, f_2 \in \mathcal{L}(V, W)$, $\forall v \in V$,

$(\lambda(f_1 + f_2))(v) = \lambda \cdot (f_1 + f_2)(v) = \lambda(f_1(v) + f_2(v))$

$= \lambda f_1(v) + \lambda f_2(v) = (\lambda f_1 + \lambda f_2)(v)$

$\Rightarrow \lambda(f_1 + f_2) = \lambda f_1 + \lambda f_2$

(ii). $\forall \lambda, \mu \in \mathbb{F}$, $\forall f \in \mathcal{L}(V, W)$, $\forall v \in V$,

$((\lambda + \mu) f)(v) = (\lambda + \mu) \cdot f(v) = \lambda f(v) + \mu f(v)$

$= (\lambda f + \mu f)(v) \Rightarrow (\lambda + \mu) f = \lambda f + \mu f$.

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Q2

30. Let

$$V = M_{2\times 2}(F), \quad W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\},$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

Proof:

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

thus $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a basis of W_1.

$\Rightarrow \dim W_1 = 3$

$$\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

thus $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis of W_2.

$\Rightarrow \dim W_2 = 2$.

Since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin W_1$, and $\dim(M_{2\times 2}(\mathbb{F})) = 4$,
so $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ with the basis of W_1 generates a basis of $M_{2\times 2}(\mathbb{F})$. Hence, $W_1 + W_2 = M_{2\times 2}(\mathbb{F})$

$\Rightarrow \dim(W_1 + W_2) = 4$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W_1 \cap W_2$, then we have $a = d = 0$, $b = -c$

that is $W_1 \cap W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in \mathbb{F} \right\} = \operatorname{span}\left(\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}\right)$

So $\dim(W_1 \cap W_2) = 1$.

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Q3

2 Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

Proof:

(" $\Leftarrow$ ") Sufficiency If $b = c = 0$, then $\forall\, p \in \mathcal{P}(\mathbb{R})$,

$$Tp = \Big(3p(4) + 5p'(6), \underbrace{\int_{-1}^{2} x^3 p(x)\,dx}\Big)$$

Since each component "$\sim$" is linear, then T is also linear.

For example, define $T_1 p = \int_{-1}^{2} x^3 p(x)\,dx$,

then ① $T_1(p+q) = \int_{-1}^{2} x^3 (p(x) + q(x))\,dx = \int_{-1}^{2} x^3 p(x)\,dx + \int_{-1}^{2} x^3 q(x)\,dx$

$= T_1 p + T_1 q$

② $T(\lambda p) = \int_{-1}^{2} x^3 \cdot \lambda \cdot p(x)\,dx = \lambda \int_{-1}^{2} x^3 p(x)\,dx = \lambda T_1 p$.

(" $\Rightarrow$ ") Necessity If T is a linear map, then $\forall\, p, q \in \mathcal{P}(\mathbb{R})$, $T(p+q) = Tp + Tq$

$$\Rightarrow \begin{cases} 3(p(4)+q(4)) + 5(p'(6)+q'(6)) + b(p(1)+q(1))(p(2)+q(2)) \\ = 3p(4) + 5p'(6) + b\,p(1)\,p(2) + 3q(4) + 5q'(6) + b\,q(1)\,q(2) \\ \int_{-1}^{2} x^3 (p(x)+q(x))\,dx + c\sin(p(0)+q(0)) \\ = \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0) + \int_{-1}^{2} x^3 q(x)\,dx + c\sin q(0) \end{cases}$$

$\Rightarrow \forall\, p, q \in \mathcal{P}(\mathbb{F})$,

$$\begin{cases} (p(1)\,q(2) + p(2)\,q(1))\,b = 0 \\ (\sin(p(0)+q(0)) - \sin p(0) - \sin q(0))\,c = 0 \end{cases}$$

So we just need to take $p(x) = \frac{\pi}{2}$, $q(x) \equiv \pi \in \mathcal{P}(\mathbb{R})$

then $\begin{cases} \pi^2 \cdot b = 0 \\ -2c = 0 \end{cases} \Rightarrow b = c = 0$

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